

Multiplicative Aspects of the Halperin-Carlsson Conjecture

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Abstract

We use the multiplicative structure of the Koszul resolution to give short and simple proofs of some known estimates for the total dimension of the cohomology of spaces which admit free torus actions and some analogous results for filtered differential modules over polynomial rings. We also point out the possibility of improving these results in the presence of a multiplicative structure on the so-called minimal Hirsch-Brown model for the equivariant cohomology of the space.

Key words: free torus actions, equivariant cohomology, Koszul complexes, minimal Hirsch-Brown model
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1 Introduction

All spaces considered in this note are assumed to be paracompact Hausdorff.

The Halperin-Carlsson conjecture is the following statement:

(HCC) If a torus $S^1 \times \cdots \times S^1$ (resp. a p -torus, $\mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$, p prime) of rank r can act freely on a finite dimensional space X , then the total dimension of its cohomology, $\sum_i \dim_k H^i(X; k) \geq 2^r$. Here k is a field characteristic 0 (resp. p).

Many authors, see e.g. [Al],[AH],[AP1],[AP2],[AB],[Ba],[Bo],[Ca1]-[Ca5],[Hl],[Hn],[Ya], have studied variants of this problem and have contributed results with respect to different aspects of the conjecture. For part of the literature and a summary of certain results see e.g. the discussion in [AD] and [AP2],(4.6.43).

It seems that in those cases where one can prove the conjecture, the multiplicative structure plays an essential role, or one considers rather special spaces like products of spheres of the same dimension.

To my knowledge the best general estimate for $\sum_i \dim_k H^i(X; k)$ is the following. We refer to the real torus case as 'case(0)', and to the p -torus as 'case(p)'.

Theorem 1.1. (a) *In case(0) the conjecture holds for $r \leq 3$, and for any $r \geq 3$ one has: $\sum_i \dim_k H^i(X; k) \geq 2(r+1)$.*

(b) *In case(2) the conjecture holds for $r \leq 3$, and for any $r \geq 3$: $\sum_i \dim_k H^i(X; k) \geq 2r$.*

(c) *In case(p), p odd, the conjecture holds for $r \leq 2$, and for any r : $\sum_i \dim_k H^i(X; k) \geq r+1$.*

In [ABI] and [ABIM] related results in a rather general algebraic context are discussed and proved.

In this note me make use of some rudiments of multiplicative structure, which come from the *dga*-structure of Koszul complexes, to give simple proofs of some of the known results

mainly in the graded context. The method of proof suggests on one hand that it might be useful to study the multiplicative structure up to homotopy on the minimal Hirsch-Brown in more detail; on the other hand the method seems flexible enough to apply also in more general algebraic situations (cf. Cor.4.6).

2 Maps between Koszul complexes

In this section k denotes a field of arbitrary characteristic. For any integer m let $K_r(m)$ be the Koszul complex corresponding to the regular sequence $(t_1^{m+1}, \dots, t_r^{m+1})$ in $R := k[t_1, \dots, t_r]$, $(\deg(t_i) = 1)$. $K_r(m) = \Lambda_R(s_1^m, \dots, s_r^m)$, is the exterior algebra over R in r generators s_1^m, \dots, s_r^m of degree m with differential $d(s_i^m) = t_i^{m+1}, d(t_i) = 0, i = 1, \dots, r$, extended as a derivation to obtain a *dga* (differential graded algebra). The Koszul complex can be considered as the minimal resolution of the R -module $R/(t_1^{m+1}, \dots, t_r^{m+1})$. We consider R -linear maps $\gamma : K_r(m) \rightarrow K_r(0)$ from the R -cochain complex $K_r(m)$ to the R -cochain complex $K_r(0)$, which lift the projection $H(K_r(m)) = R/(t_1^{m+1}, \dots, t_r^{m+1}) \rightarrow H(K_r(0)) = R/(t_1^1, \dots, t_r^1) \cong k$. But we do not assume that the maps preserve the length of exterior products nor the grading, i.e. they are just morphisms of the underlying differential modules. By $\text{rk}(\gamma)$ we mean the rank of the map induced by γ on the localized modules, inverting all non-zero homogenous polynomials in R .

Lemma 2.1. (a) *If γ is a dga-map (in particular multiplicative), then $\text{rk}(\gamma) = 2^r$. (cf. [BE], Prop.1.4.)*
 (b) *For any γ as above $\text{rk}(\gamma) \geq 2r$.*

Proof: Since $K_R(0)$ is a free resolution of k , considered as a R -module via the canonical augmentation, which maps all the t_i 's to 0, any two maps of the form γ are homotopic. In particular any such γ is homotopic to the map ι defined as the dga-map which sends s_i^m to $t_i^m s_i^0$. Hence γ induces the same map as ι in cohomology for any coefficients. We consider the induced map in cohomology with coefficients in $\bar{R} := R/(t_1^{m+1}, \dots, t_r^{m+1})$ and use the map ι for calculations. The induced boundary on $K_r(m) \otimes \bar{R}$ vanishes and it is easy to see, that $H(K_R(0) \otimes \bar{R})$ is the exterior algebra over k , generated by $t_1^m s_1^0, \dots, t_r^m s_r^0$ (Note that the superscript for s_i is used just to identify the element, whereas the superscript for t_i denotes an exponent.) The induced map $\gamma^* = \iota^* : K_r(m) \otimes \bar{R} \rightarrow H(K_R(0) \otimes \bar{R})$ maps t_i to $[t_i^m s_i^0]$ and is multiplicative. In particular the element $s_{1\dots r}^m := s_1^m \wedge \dots \wedge s_r^m$ is mapped to the non-zero element $[t_1^m \dots t_r^m s_{1\dots r}^0]$. Therefore $\gamma(s_{1\dots r}^m)$ is non-zero in $K_R(0)$. If γ is assumed to be multiplicative then it follows that it must be injective for the following reason: For any element $x \in K_r(m)$ there exists an element $y \in K_r(m)$ such that $xy = qs_{1\dots r}^m$, with $q \in R$ (because the exterior algebra fulfills Poincaré duality over the quotient field). So for $x \neq 0$, $\gamma(x)$ must be non-zero, otherwise $\gamma(xy)$ would vanish, which is not the case since $\gamma(s_{1\dots r}^m)$ is non-zero in $K_R(0)$. Hence we get part(a) of the lemma, since localization is exact.

To prove part(b) we change γ by first restricting the map to the free R -submodule $\Lambda^{\leq 1}$ of $K_r(m)$ generated by $1, s_1^m, \dots, s_r^m$ and then extending it multiplicatively. On this submodule the two maps coincide and are injective by part(a). Since γ commutes with the respective boundaries it is also non-zero on all elements in $K_r(m)$, which are mapped by the boundary into non-zero elements of $\Lambda^{\leq 1}$. An R -linear combination of the elements $1, s_1^m, \dots, s_r^m, s_{12}^m, \dots, s_{2r}^m$ is mapped to a non-zero element in $\Lambda^{\leq 1}$ if at least one of the coefficients of the s_{1j}^m is non-zero. Hence γ is injective on the free R -module generated by the $2r$ above elements. This gives part(b) of the Lemma. \square

Some time ago Martin Fuchs found an example of a map γ with $rk(\gamma) < 2^r$, and $r = 4$. The following example shows that one can not improve the estimate in part(b) of the above lemma for $r = 3$ without additional assumptions.

Example 2.2. The following map $\gamma : K_3(1) \longrightarrow K_3(0)$ is homotopic to the standard map ι and has rank equal to $2r = 6$, ($k = \mathbb{F}_2$). Define $\gamma = \iota + dh + hd$ with $h(s_1^1) = s_{123}^0$, $h(s_{23}^1) = t_3 s_{12}^0$ and otherwise equal to 0 for the standard basis of $K_3(1)$. Direct computation shows that γ vanishes on $x := t_1 t_3 s_{12}^1 + t_2 s_{123}^1$ and on dx , and x and dx are linearly independent over R . Hence $rk(\gamma) \leq 6$ and by the above lemma it must be indeed equal to 6. \square

In view of the topological applications, (see Section 5), we will also consider the degree convention $\deg(t_i) = 2$ in case (0); Lemma 2.1. holds in this context, too.

The above example (with appropriate signs) also works for other fields if $\deg(t_i) = 1$, but it does not work, if one puts $\deg(t_i) = 2$ (cf. Theorem 5.1.(b)).

For later use we prove the following technical result:

Lemma 2.3. *In case (0), (i.e. $\deg(t_i) = 2$, $\deg(s_i^m) = (2m + 1)$), if $r \geq 3$ then any map γ as above, which preserves degrees, is injective on the free R -submodule of $K_r(m)$, which is generated by the elements $s_1^m, \dots, s_r^m, s_{123}^m$.*

Proof: Let $x = qs_{123}^m + \sum_1^3 q_i s_i^m$ with $q, q_i \in R$ be a homogeneous element in $K_r(m)$. Note that we already know from the above arguments, that we only need to consider the case $q \neq 0$. Let J be the ideal generated by $qt_i^{m+1}, i = 1, \dots, r$ and $q_i t_i^{m+1}, i = 1, 2, 3$. We consider cohomology with coefficients in R/J , so x becomes a cycle; and we show that ι^* and hence γ^* is non-zero on $[x]$, which, of course implies that $\gamma(x) \neq 0$. The maps ι sends the element x to $qt_1^m t_2^m t_3^m s_{123}^0 + \sum_1^3 q_i t_i^m s_i^0$. Since $\deg(s_i^m) = 2m + 1$ in case (0), one has $\deg(q_i) = \deg(q) + 4m + 2$. So $\deg(qt_1^m t_2^m t_3^m) < \deg(q_i t_i^{m+1})$, and since $qt_1^m t_2^m t_3^m$ is not contained in the ideal generated by $qt_i^{m+1}, i = 1, \dots, r$, it is also not contained in J . Hence $[\iota(x)]$ is non-zero in $H^*(K_r(0) \otimes_R R/J)$. \square

Lemma 2.3. also holds (for any characteristic) if $\deg(t_i) = 1$, but we will apply it later in the above form.

3 Minimal models

We give a brief account of some facts about (additive) minimal models of cochain complexes over the graded polynomial ring $R = k[t_1, \dots, t_r]$, where k is a field of arbitrary characteristic. The (additive) minimal model here plays a role similar to that of the minimal resolution in homological algebra. We do not consider any product structure (besides the R -module structure) on the complexes in this section. So these minimal models should not be confused with the (multiplicative) Sullivan minimal models. The material is known, see e.g. [AP2], Appendix B, and can also be drawn from several other sources which sometimes deal with much more general situations, but we hope that it will be convenient for the reader to get a short and rather elementary presentation of what is needed in this note.

Let \tilde{C} be a free cochain complex over R with boundary \tilde{d} of total degree 1. As R -module, $\tilde{C} \cong C \otimes R$, where C is a k -vector space, and the unspecified tensor product, \otimes , is taken over the field k . We want to define a free cochain complex, which is homotopy equivalent to the given one and has minimal rank over R . Such a complex turns out to be unique up to isomorphism and is called the (additive) minimal model of \tilde{C} .

The R -linear boundary \tilde{d} can be written as a sum $d + d'$, with d a boundary on $C \otimes k$, and $d'(\tilde{C}) \subset C \otimes I$, where I denotes the augmentation ideal of R . If we consider the elements in \tilde{C} as polynomials in the variables $t_i, i = 1, \dots, r$ with coefficients in C , the part d of the boundary correspond to the coefficients of 1 i.e. the constant part of \tilde{d} , and d' to the higher terms in the t_i 's.

$(C, d) \cong (\tilde{C} \otimes_R k, \tilde{d} \otimes_R k)$, where k is an R -module via the augmentation, is a cochain complex over the field k . Hence we can write C as a (non-canonical) direct sum $H \oplus B \oplus D$, where $H = H(C, d)$, and d corresponds to an isomorphism (also called d) from D to B . We extend this isomorphism R -linearly to $B \otimes R$ and obtain this way a contractible cochain complex $\tilde{N} \cong (B \oplus D) \otimes R$. We imbed this complex into $C \otimes R$, imbedding $B \otimes R$ by R -linear extension of the embedding of D into the direct sum above and extending this map to be compatible with the respective boundaries. Note that this is well defined since the boundary in \tilde{N} maps $D \oplus R$ isomorphically to $B \otimes R$, but it is not(!) the R -linear extension of the embedding of $B \oplus D$, if d' is non-zero. Nevertheless the map defined is an embedding of a contractible cochain complexes onto a direct summand as an R -module, and the quotient complex, \tilde{H} , is isomorphic as an R -module to $H \otimes R$, but inherits a twisted boundary \tilde{d} . If one tensors $\tilde{C} \cong \tilde{H} \oplus \tilde{N}$ with k , considered as a R -module via the augmentation (or in other words: if one restricts to constant terms with respect to the variables t_i), then one gets back the direct sum decomposition given above. In particular the constant part of the boundary in \tilde{H} is zero. The complex \tilde{H} is R -homotopy equivalent to \tilde{C} , since it is obtained from the latter by dividing out a contractible direct summand. It is not difficult to see that up to isomorphism of R -complexes there is only one free R -complex which is R -homotopy equivalent to \tilde{C} , and has a boundary with vanishing constant term (i.e. which vanishes when tensored with k over R). The complex \tilde{H} is called the (additive) minimal model of the complex \tilde{C} . It follows from $\tilde{d} \circ \tilde{d} = 0$ that the part of \tilde{d} , which is linear in the t_i 's, anti-commutes with d . Therefore this linear part induces a map on $H(C, d)$ which is in fact the linear part of the boundary of \tilde{H} . For every index i we define a map λ_i from H to itself by assigning to an element $x \in H$ the coefficient of t_i in $\tilde{d}(x)$. Since $\tilde{d} \circ \tilde{d} = 0$, the maps λ_i anti-commute for $i, j = 1, \dots, r$. Hence they define an action of the exterior algebra $\Lambda_k(\lambda_1, \dots, \lambda_r)$ on H . Let Λ^+ be the ideal generated by $\lambda_1, \dots, \lambda_r$. The length, $\ell_\Lambda(H^q)$, of H^q as a $\Lambda_k(\lambda_1, \dots, \lambda_r)$ - module is definite as the minimal integer i , such that $(\Lambda^+)^i H^q = 0$. One has proper inclusions $(\Lambda^+)^i H^q \supset (\Lambda^+)^{i+1} H^q$ for $i = 0, \dots, (\ell_\Lambda(H^q) - 1)$.

Assume now that $\dim_k H$ is non-zero and finite. We define a filtration on the minimal model \tilde{H} by subcomplexes inductively in the following way (cf.[AP2],Sec.1.4):

Definition 3.1. Let $\tilde{H} \cong H \otimes R$ be a minimal model as above. We define

$$\mathcal{F}_0(H) = 0 \text{ and } \mathcal{F}_0(\tilde{H}) = 0,$$

$$\mathcal{F}_1(H) := \ker(\tilde{d} : H \cong H \otimes k \longrightarrow \tilde{H}) \text{ and } \mathcal{F}_1(\tilde{H}) := \mathcal{F}_0(H) \otimes R.$$

Let us assume that $\mathcal{F}_i(\tilde{H}) := \mathcal{F}_i(H) \otimes R$ is already defined, then we put

$$\mathcal{F}_{i+1}(H) := \tilde{d}^{-1}(\mathcal{F}_i(\tilde{H})) \cap (H \otimes k) \text{ and } \mathcal{F}_{i+1}(\tilde{H}) := \mathcal{F}_{i+1}(H) \otimes R.$$

The length of the filtration, $\ell(\mathcal{F}_*(\tilde{H}))$, is the smallest index i for which $\mathcal{F}_i(\tilde{H})$ and $\mathcal{F}_{i+1}(\tilde{H})$ coincide.

We summarize some of the properties of this filtration in the following proposition.

Proposition 3.2. (a) One has proper inclusions of free subcomplexes

$$\mathcal{F}_0(\tilde{H}) = 0 \subset \mathcal{F}_1(\tilde{H}) \cdots \subset \mathcal{F}_{\ell(\mathcal{F}_*(\tilde{H}))}(\tilde{H}) = \tilde{H}, \text{ which are direct summands as } R\text{-modules.}$$

$$(b) \tilde{d}(\mathcal{F}_i(\tilde{H})) \subseteq \mathcal{F}_{i-1}(\tilde{H}), \text{ for } i = 0, \dots, \ell(\mathcal{F}_*(\tilde{H}))$$

(c) The complex \tilde{H} admits an augmentation $\varepsilon : \tilde{H} \longrightarrow k$ compatible with the respective boundaries (where the boundary on k is trivial), such the restriction to $\mathcal{F}_0(\tilde{H})$ is surjective.

(d) The boundary $\tilde{d} : \mathcal{F}_i(\tilde{H}) \longrightarrow \mathcal{F}_{i-1}(\tilde{H})$ induces a non-trivial map from $(\mathcal{F}_i(\tilde{H})) / \mathcal{F}_{i-1}(\tilde{H})$ to $(\mathcal{F}_{i-1}(\tilde{H})) / \mathcal{F}_{i-2}(\tilde{H})$ for $i = 1, \dots, \ell(\mathcal{F}_*(\tilde{H}))$.

$$(e) \dim_k H \geq \sum_q \ell_\Lambda(H^q) \geq \ell(\mathcal{F}_*(\tilde{H}))$$

(f) If the action of Λ on H is trivial, i.e. the terms in \bar{d} , which are linear with respect to the t_i 's vanish, then $\#\{q; H^q \neq 0\} \geq \ell(\mathcal{F}_*(\tilde{H}))$.

Proof: The properties (a)-(e) follow directly from the definition using in particular the following facts:

- the total degree of the boundary is 1,
- the constant part vanishes,
- the linear part of the boundary of $(\Lambda^+)^i H^q \otimes k$ is contained in $(\Lambda^+)^{(i+1)} H^q \otimes R$, and the higher order parts are contained in $H^{<q} \otimes R$.

Part (f) follows immediately from the fact that, under the assumption made, the boundary of $H^i \otimes k$ is contained in $H^{\leq(i-1)} \otimes R$ for degree reasons. \square

The boundary of \tilde{H} induces a boundary on the associated, graded complex of the filtration \mathcal{F}_* , which is non-trivial for all $(\mathcal{F}_i(\tilde{H})/\mathcal{F}_{i-1}(\tilde{H}))$, $i = 0, \dots, \ell(\mathcal{F}_*)$. Since the composition of two successive boundary maps vanishes, one has $rk(\mathcal{F}_i(\tilde{H})/\mathcal{F}_{i-1}(\tilde{H})) \geq 2$ for $i = 1, \dots, (\ell(\mathcal{F}_*) - 1)$, otherwise one of the two maps in the composition would have to vanish, which is not the case, see Prop.3.2.(d). Hence we get(cf.[AP2],Cor.(1.4.21)):

Corollary 3.3. $\dim_k(H^*) \geq 2(\ell(\mathcal{F}_*(\tilde{H})) - 1)$.

4 Factorization

In this section we combine the results of two previous sections. We consider free cochain complexes \tilde{C} over the ring R as in the previous section. We assume that \tilde{C} has an augmentation $\varepsilon : \tilde{C} \rightarrow k$, which induces a surjection in cohomology.

Proposition 4.1. *Let \tilde{C} be a cochain complex over R as above and such that $H^{\geq(m+1)}(\tilde{C})$ vanishes, then there exists a map of R -complexes $\alpha : K_r(m) \rightarrow \tilde{C}$, such that $\varepsilon^* \alpha^* : H(K_r(m)) = R/(t_1^{m+1}, \dots, t_r^{m+1}) \rightarrow k$ is the canonical projection.*

The proof of this proposition is by standard homological algebra. One defines α inductively over the lengths of exterior products in $K_r(m)$. The assumption on the vanishing of the cohomology in high degrees allows to choose the images of the elements s_i^m compatible with the boundary, and so on. Cf. e.g. [AP2],Lemma (1.4.17). \square

Let \tilde{C} be free differential R -module with a filtration $\mathcal{F}_*(\tilde{C})$, which has the properties (a) and (b) of Prop.3.2.(for \tilde{C} in place of \tilde{H}). In [ABI] such an object is called a free differential flag. We assume in addition that \tilde{C} has an surjective augmentation $\varepsilon : \mathcal{F}_0(\tilde{C}) \rightarrow k$, which extends to morphism of differential modules $\varepsilon : \tilde{C} \rightarrow k$. We consider $K_r(0)$ with the filtration by length of exterior products and with the canonical augmentation.

Proposition 4.2. *Under the above assumptions there exists a morphism of differential R -modules $\beta : \tilde{C} \rightarrow K_r(0)$, which commutes with the respective augmentations.*

Again the proof is by standard homological algebra using induction over the filtration degree. See [Ba], cf. [AP2], Lemma (1.4.18)), for the corresponding result in the presence of an additional grading. \square

Corollary 4.3. *If \tilde{C} is a free cochain complex over the graded polynomial ring $k[t_1, \dots, t_r]$, k a field of arbitrary characteristic, $\deg(t_i) = 1$, such that $\dim_k H^*(\tilde{C})$ is non-zero finite, then*

(a) $\dim_k H = \sum_i H^i(\tilde{C} \otimes k) \geq 2r$,

(b) *For the length of the filtration on the minimal model of \tilde{C} one has:*

$$\ell(\mathcal{F}^*(\tilde{H})) \geq \sum_q \ell_{\Lambda}(H^q) \geq (r+1).$$

(c) If the linear part of the boundary on the minimal model vanishes, one has: $H^q(\tilde{C})$ is non zero for at least $r+1$ degrees q .

Proof: Part(a): By the two above propositions, applied to the minimal model \tilde{H} of \tilde{C} , one obtains morphisms α and β such that the composition $\beta\alpha$ sends $1 \in K_r(m)$ to $1 \in K_r(0)$. So, by the Lemma 2.1, the rank of this composition is greater or equal to $2r$. Since the map factors through the minimal model, \tilde{H} , the rank of the model (when localized) must also be $\geq 2r$. But as an R -module this model is free of dimension $\dim_k H = \sum_q H^q(\tilde{C} \otimes k)$. Hence the assertion follows.

Part(b): The length of the filtration by exterior products on $K_r(0)$ is equal to $(r+1)$. The element $s_{1\dots r}^0$ has filtration length r . The multiple $t_1^m \cdots t_r^m s_{1\dots r}^0$ represents a non-zero element in $H^*(K_r(0) \otimes_R R/J)$. This class is in the image of ι^* and hence also in the image of β^* (see Section 2). Since by Prop. 4.2. the map β can be chosen to preserve filtrations, it follows that the length of any filtration of \tilde{H} has to be strictly greater than r . Together with Prop.3.2.(e) one obtains the assertion.

Part(c) follows from Prop.3.2.(f). □

Remark 4.4. Instead of using Lemma 2.1 one can deduce part(a) of the above corollary from part(b) and Cor.3.3.

Remark 4.5. There are far reaching recent generalizations, which give similar bounds in a much more general algebraic context (see [ABI], [ABIM]). Our main point here is to present a rather elementary proof in the most classical, graded situation. But the method has also some potential to be applied more generally, see Cor.4.6. On the other hand it is rather doubtful that it can lead to better estimates without substantial additional effort, as the above Example 2.2. shows.

Let (\tilde{C}, \tilde{d}) be an (ungraded) free differential R -module ($R = k[t_1, \dots, t_r]$ also considered without grading) with a filtration and an augmentation as above. We assume that the filtration is minimal, i.e. that also Prop.3.2.(d)(for \tilde{C} in place of \tilde{H}) holds. Finally we suppose that the annihilator ideal of $H^*(\tilde{C})$ contains the elements $t_1^{m+1}, \dots, t_s^{m+1}$. This is a replacement for the assumption that $H^{\geq(m+1)}(\tilde{C})$ vanishes in the graded case.

Corollary 4.6. Under the above assumptions one has:

- (a) $\text{rk}_R(\tilde{C}) \geq 2s$.
- (b) The length of the filtration of \tilde{C} is at least $s+1$.

The proof is completely analogous to that of Cor.4.3.(a) and (b). The assumptions above allow to obtain a factorization up to homotopy of the standard map $\iota : K_s(m) \longrightarrow K_s(0)$ through the filtered differential module \tilde{C} using the Propositions 4.1. and 4.2., more precisely: The assumption on the annihilator ideal allows to apply Proposition 4.1. adapted to the situation at hand, and the assumptions on the augmentation make sure that one can apply Proposition 4.2. (We shift from $k[t_1, \dots, t_r]$ -modules to $k[t_1, \dots, t_s]$ -modules via the canonical inclusion and projection.) □

The following corollary is of a somewhat different nature.

Corollary 4.7. If \tilde{C} is as in Cor.4.3, then

$\dim_k H(\tilde{C} \otimes_R R/(t_1^{m+1}, \dots, t_r^{m+1})) \otimes_R R/(t_1, \dots, t_r) \geq 2^r$, for large enough m . In other words: The minimal number of generators of $H(\tilde{C} \otimes_R R/(t_1^{m+1}, \dots, t_r^{m+1}))$ as an R -module is greater or equal to 2^r .

Proof: The map $\iota^* : H^*(K_r(m) \otimes_R R/(t_1^{m+1}, \dots, t_r^{m+1})) \longrightarrow H^*(K_r(0) \otimes_R R/(t_1^{m+1}, \dots, t_r^{m+1}))$ maps $[t_i]$ to $[t_i^m s_i^0]$, see proof of Lemma 2.1. Taking the tensor product of this map with $R/(t_1, \dots, t_r)$ over R one gets an isomorphism. Since the map factors through $H(\tilde{C} \otimes_R R/(t_1^{m+1}, \dots, t_r^{m+1})) \otimes_R R/(t_1, \dots, t_r)$, the assertion follows. \square

Note that the above corollary also applies to the minimal model \tilde{H} of \tilde{C} . So $\dim_k H(\tilde{H} \otimes_R R/(t_1^{m+1}, \dots, t_r^{m+1})) \otimes_R R/(t_1, \dots, t_r) \geq 2^r$. But the Halperin-Carlsson conjecture in this algebraic context can be stated as $\dim_k (H^*(\tilde{H} \otimes_R R/(t_1, \dots, t_r))) = \tilde{H} \otimes_R R/(t_1, \dots, t_r) \geq 2^r$. Although these two statements look rather similar, they differ by an interchange of taking homology and tensor product.

5 Applications to torus actions

In this section we give applications of our previous results to free torus (resp. 2-torus) actions. We then have $H^*(BG; k) \cong R = k[t_1, \dots, t_r]$, where $\text{char}(k) = 0$ and $\deg(t_i) = 2$ (resp. $\text{char}(k) = 2$ and $\deg(t_i) = 1$). Let X be a finite dimensional space on which a torus, $S^1 \times \dots \times S^1$ (resp. a 2-torus, $\mathbb{Z}/2 \times \dots \times \mathbb{Z}/2$) of rank r acts freely. We will use Cor.4.3. to get estimates for the size of the cohomology of X with coefficient in a field k of characteristic zero (resp. 2). We consider the equivariant cohomology of the space X and briefly recall some facts about the minimal Hirsch-Brown model of the equivariant cohomology for a G -space X (see [AP2] for details). For a G -space X the Borel construction gives a fibration $X \longrightarrow X_G := EG \times_G X \longrightarrow BG$ where BG is the classifying space and EG the universal (free, contractible) G -space. For $G = (S^1)^r$ (resp. $(\mathbb{Z}/2)^r$) there is the following additive minimal cochain model over $R \cong H^*(BG; k)$ for the cohomology of X_G , the so-called minimal Hirsch-Brown model (s.[AP2]): $H^*(X; K) \tilde{\otimes} H^*(BG; k)$, where the tilde indicates that the tensor product carries a twisted differential, which in a sense reflects the G -action on the cochain level (cf.[AP2]).

In case $G = (\mathbb{Z}/p)^r$ the minimal Hirsch-Brown model is in general not a cocchain complex over $H^*(BG; k)$, but only a module over the polynomial part of $H^*(BG; k)$. The behavior with respect to the exterior part of $H^*(BG; k)$ is rather complicated and our algebraic results above do not suffice to give the results stated in the introduction for p -tori, p odd. We refer to [Ba] and [AP2],(1.4.14) for more involved proofs of these results in a similar spirit.

If G acts freely on X then $X_G \simeq X/G$; in particular

$$H^*(X_G; k) = H^*(H^*(X; k) \tilde{\otimes} H^*(BG; k)) \cong H^*(X/G; k)$$

In the case $G = (S^1)^r$ the last isomorphism even holds under the weaker assumption that all isotropy groups are finite. If, in addition, X is finite dimensional then so is X/G , and hence $H^i(X_G; k) \cong H^i(X/G; k) = 0$ for large enough i , say $i \geq 2(m+1)$ in case $G = (S^1)^r$ (resp. $i \geq m+1$ in case $G = (\mathbb{Z}/2)^r$).

We apply Cor.4.3 and Lemma 2.3 to the minimal Hirsch-Brown model to obtain part of the result stated in the introduction.

Theorem 5.1. (a) *If X is a finite dimensional space on which a 2-torus $G = (\mathbb{Z}/2)^r$ acts freely, then $\dim_k \sum_i H^*(X; k) \geq 2r$.*
 (b) *$\sum_q \ell_\Lambda(H^q(X; k)) \geq r+1$; in particular, if the action on X induces the trivial action in cohomology, then $H^i(X; k)$ is non-zero for at least $(r+1)$ degrees i ($\text{char}(k) = 2$).*
 (c) *If X is a finite dimensional space on which a torus $G = (S^1)^r$ acts almost freely, then $\dim_k \sum_i H^*(X; k) \geq 2r$. The number of degrees, for which $H^i(X; k)$ is non-zero is at least*

$r + 1$.

For $r \geq 3$, $\dim_k \sum_i H^*(X; k) \geq 2(r + 1)$ ($\text{char}(k) = 0$).

Proof: Part(a) and the first two parts of (b) follow immediately from Cor.4.3. (Note that trivial action in cohomology implies that the linear part of the boundary of the minimal Hirsch-Brown model vanishes in case(2); for case(0) this is always true). To show the slightly improved inequality in part(b) we observe that, in case(0), \tilde{H} when localized (by inverting all non-zero homogeneous polynomial in R) inherits a $\mathbb{Z}/2\mathbb{Z}$ -grading by even and odd degree. Since the localized cohomology vanishes, the ranks of the even and the odd part in \tilde{H} are equal. Now Lemma 2.3 together with the above factorization shows that the odd part of \tilde{H} has rank at least $r+1$. So $\text{rk}(\tilde{H}) = \dim_k \sum_i H^i(X; k) \geq 2(r + 1)$. \square

The minimal Hirsch-Brown model carries a multiplicative structure which induces the cup product in equivariant cohomology. In general the multiplication on the model is commutative and associative only up to (higher) homotopies. In rather special cases, e.g. in case(0) when X is a product of spheres of odd dimension, the minimal Hirsch-Brown model coincides with the Sullivan minimal model of the Borel construction X_G , and hence is a differential graded algebra (dga). In such a situation one can derive from the above results the following corollary.

Corollary 5.2. *If the minimal Hirsch-Brown model of the finite dimensional, free G -space X carries a dga-structure (over R), then $\dim_k \sum_i H^*(X; k) \geq 2^r$ (cf.[BE], Prop.1.4)*

Proof: Under the given hypothesis the map $\alpha : K_r(m) \longrightarrow \tilde{H}$ can be chosen to multiplicative, and hence an argument analogous to that given for Lemma 2.1 (a) shows that α must be injective. Therefore $\text{rk}(\tilde{H}) = \dim_k \sum_i H^*(X; k) \geq 2^r$. \square

Remark 5.3. The above results are analogous to results in homological algebra using a multiplicative structure (if it exists (!)) on the minimal resolution of a finite module over a polynomial ring (see [BE],[Av]). As in the latter case it would suffice for the result of the corollary, to assume that \tilde{H} carries an (associative) dg-module structure over $K_r(m)$ (cf.[Av], Prop.6.4.1). Unfortunately we do not know of any new examples to which the above corollary could be applied, but it might be interesting to study the question of existence of such module structures for special classes of spaces, e.g. (rationally) formal spaces in the sense of Sullivan.

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References

- [AB] Adem, A., Browder, W.: *The free rank of symmetry of $(S^n)^k$* , Invent.Math.92(1988), 431-440.
- [AD] Adem, A., Davis, J.F.: *Topics in transformation groups*, Handbook of Geometric Topology, 1-54, North-Holland, Amsterdam (2002).
- [Al] Allday, C.: *The rank of a space*, Trans. Amer.Math. Soc. 166(1972), 173-185
- [AH] Allday, C., Halperin, S.: *Lie group actions on spaces of finite rank*, Quart. J. Math. Oxford (2) 29(1978), 69-76
- [AP1] Allday, C., Puppe, V.: *Bounds on the torus rank*, Poznan 1985, Proceedings, Springer Lecture Notes in Math. 1217(1986), 1-10.

[AP2] Allday, C., Puppe, V.: *Cohomological methods in transformation groups*, Cambridge University Press (1993).

[Av] Avramov, L.L.: *Obstructions to the existence of multiplicative structures on minimal free resolutions*, Amer.J.Math. 103(1981), 1-31.

[ABI] Avramov, L.L., Buchweitz, R.-O., Iyengar, S.B.: *Class and rank of differential modules*, Invent. Math. 169(2007), 1-35.

[ABIM] Avramov, L.L., Buchweitz, R.-O., Iyengar, S.B., Miller, C.: *Homology of perfect complexes*, arXiv: math/0609008v2 (2007)

[Ba] Baumgartner, C.: *On the cohomology of free p -torus actions*, Archiv Math. 61(1993), 137-149.

[Bo] Browder, W.: *Cohomology and group actions*, Invent. Math 71(1983), 599-607

[BE] Buchsbaum, D.A., Eisenbud, D.: *Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3*, Amer.J.Math. 99(1977), 447-485.

[Ca1] Carlsson, G.: *On the non-existence of free actions of elementary abelian groups on products of spheres*, Amer. J. Math. 102(1980), 1147-1157.

[Ca2] Carlsson, G.: *On the rank of abelian groups acting freely on $(S^n)^k$* , Invent. Math. 69(1982), 393-400.

[Ca3] Carlsson, G.: *On the homology of finite free $(\mathbb{Z}/2)^n$ -complexes*, Invent. Math. 74(1983), 139-147.

[Ca4] Carlsson, G.: *Free $(\mathbb{Z}/2)^k$ -actions and a problem in commutative algebra*, Poznan 1985, Proceedings, Springer Lecture Notes in Math. 1217(1986), 79-83.

[Ca5] Carlsson, G.: *Free $(\mathbb{Z}/2)^3$ -actions on finite complexes*, (Princeton N.J. 1983), Ann. of Math. Studies 113(1987), 332-344.

[Hl] Halperin, S.: *Rational homotopy and torus actions*, Lond. Math. Soc. Lect. Note Ser. 93(1985), 293-305.

[Hn] Hanke, B.: *The Stable Rank of Symmetry of $S^{n_1} \times \dots \times S^{n_k}$* , preprint (2008).

[Ya] Yalçin, E.: *Group actions and group extensions*, Trans. Amer. Math.Soc. 352(2000), 2689-2700.

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